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# Bounded Mean Oscillation and Bandlimited Interpolation in the Presence of Noise

Gaurav Thakur*

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**Abstract**

We study some problems related to the effect of bounded, additive sample noise in the bandlimited interpolation given by the Whittaker-Shannon-Kotelnikov (WSK) sampling formula. We establish a generalized form of the WSK series that allows us to consider the bandlimited interpolation of any bounded sequence at the zeros of a sine-type function. The main result of the paper is that if the samples in this series consist of independent, uniformly distributed random variables, then the resulting bandlimited interpolation almost surely has a bounded global average. In this context, we also explore the related notion of a bandlimited function with bounded mean oscillation. We prove some properties of such functions, and in particular, we show that they are either bounded or have unbounded samples at any positive sampling rate. We also discuss a few concrete examples of functions that demonstrate these properties.

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## 1 Introduction

The classical Whittaker-Shannon-Kotelnikov (WSK) *sampling theorem* is a central result in signal processing and forms the basis of analog-to-digital and digital-to-analog conversion in a variety of contexts involving signal encoding, transmission and detection. If we normalize the Fourier transform as $\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-2\pi i\omega t}dt$, then the sampling theorem states that a function $f \in L^2(\mathbb{R})$ with $\mathrm{supp}(\hat{f}) \subset [-\frac{b}{2}, \frac{b}{2}]$ can be expressed as a series of the form

$$f(t) = \sum_{k=-\infty}^{\infty} a_k \frac{\sin(\pi(bt-k))}{\pi(bt-k)}, \tag{1}$$

where $a_k = f(k)$ are its samples. Conversely, for a given collection of data $\{a_k\} \in l^2$, the series (1) defines a function in $L^2(\mathbb{R})$ with $\mathrm{supp}(\hat{f}) \subset [-\frac{b}{2}, \frac{b}{2}]$ called the *bandlimited interpolation* of $\{a_k\}$.

*Program in Applied and Computational Mathematics, Princeton University, Princeton, NJ 08544, USA, email: gthakur@princeton.edu

The calculation or approximation of this series is a standard procedure in many applications. For example, in audio processing it is used for resampling signals at a higher rate, typically by applying a lowpass filter to the piecewise-constant zero order hold function of the samples [6]. In this paper, we consider the situation of bounded noise in the samples $a_k$. Building on recent work by Boche and Mönich on related problems [3, 4, 5], we study some properties of the effect of the noise on the bandlimited interpolation $f$.

Before we discuss our problems, it will be convenient to define the *Paley-Wiener spaces* for $1 \leq p \leq \infty$ by

$$PW_b^p = \left\{ f \in L^p : \operatorname{supp}(\hat{f}) \subset \left[ -\frac{b}{2}, \frac{b}{2} \right] \right\},$$

where $\hat{f}$ is interpreted in the sense of tempered distributions. Our notation $PW_b^p$ essentially follows Seip [12], and is slightly different from the one used by Boche and Mönich. Without loss of generality, we will set $b = 1$ in what follows.

Returning to the series (1), we consider corrupted samples of the form $a_k = T_k + N_k$, where $T_k$ are the true samples and $N_k$ is some form of noise, and we correspondingly write $f(t) = T(t) + N(t)$. One obstacle we face is that the noise $\{N_k\}$ may not naturally decay in time alongside the signal, and even if $\{T_k\} \in l^2$, it is often more physically meaningful to consider $\{N_k\} \in l^\infty$. The WSK sampling theorem shows that for any collection of samples $\{a_k\} \in l^2$, there exists a unique function $f \in PW_1^2$ with $f(k) = a_k$. However, for bounded samples $\{a_k\} \in l^\infty$, the series (1) does not necessarily converge. In fact, a given $\{a_k\} \in l^\infty$ may correspond to multiple functions $f \in PW_1^\infty$, or to no such function [3].

A simple example of the former possibility (non-uniqueness) is given by $a_k \equiv 0$, which corresponds to the functions $f(t) \equiv 0$ and $f(t) = \sin(\pi t)$. It turns out that adding one extra sample to the collection $\{a_k\}$ resolves this ambiguity, and allows us to consider the unique bandlimited interpolation of any bounded data $\{a_k\} \in l^\infty$. We discuss the details of this procedure in Section 3. The latter possibility (non-existence) is less obvious, but in [3], Boche and Mönich presented an explicit example of this phenomenon. They showed that for the samples given by $a_k = 0$, $k < 1$, and $a_k = (-1)^k / \log(k+1)$, $k \geq 1$, there is no $f \in PW_1^\infty$ with $f(k) = a_k$. It is also possible to construct other, similar examples using standard special functions, and we describe one such sequence of $\{a_k\}$ in Section 3 and discuss its properties.

The main observation of this paper is that such examples of $\{a_k\}$ are in a sense "highly oscillating." By assuming that the noise $N_k$ is statistically incoherent and defining $N(t)$ carefully, we can rule out these examples and obtain sharper statements on the behavior of $N(t)$. More precisely, we show in Section 4 that if $N_k$ is a uniformly distributed, independent white noise process, then $\sup_{r>0} \frac{1}{2r} \int_{-r}^{r} |N(t)| dt < \infty$ almost surely. In other words, the average of $|N(t)|$ is globally bounded. We find that this result does not generally hold for $\{N_k\} \in l^\infty$ that lack such a statistical condition, and we discuss examples that illustrate the differences.

We also study a second topic motivated by further understanding $N(t)$. As discussed in [7], the WSK series (1) can be interpreted as a discrete Hilbert transform operator $H$, mapping a space

of samples into a space of bandlimited functions (see also [1] and [11]). The Plancherel formula shows that $H$ maps $l^2$ into $PW_1^2$. In fact, $H$ also maps $l^p$ into $PW_1^p$ for any $1 < p < \infty$, and the series (1) converges for any $\{a_k\} \in l^p$ [10]. This can be compared with the continuous Hilbert transform, and more generally any Calderon-Zygmund singular integral operator, which maps $L^p$ into itself for any $1 < p < \infty$. Such operators behave differently for $p = \infty$, mapping $L^\infty$ into the space $BMO$ of functions with *bounded mean oscillation* [13].

It is thus reasonable to expect that if we consider samples $\{a_k\} \in l^\infty$, the "right" target space for $H$ may be one of bandlimited functions lying in the space $BMO$. However, this heuristic reasoning turns out to be incorrect. We consider bandlimited $BMO$ functions in Section 5 and establish some of their properties. In particular, we find that such a function $f$ is either in $L^\infty$ or that its samples $\{f(\frac{k}{s})\}$ are unbounded for any sampling rate $s > 0$. We exhibit a concrete example of such a function, and study it in the context of our other results.

We review some existing theory on bandlimited functions and the space $BMO$ in Section 2, and discuss some preliminary results in Section 3. The main results of the paper are presented in Sections 4 and 5. We also develop our results for a class of general, nonuniformly spaced interpolation points, given by zeros of *sine-type functions*. The above discussion for uniformly spaced points is a special case.

# 2 Background Material

We will write $f_1 \lesssim f_2$ if the inequality $f_1 \le C f_2$ holds for a constant $C$ independent of $f_1$ and $f_2$. We define $f_1 \gtrsim f_2$ similarly, and write $f_1 \approx f_2$ if both $f_1 \lesssim f_2$ and $f_1 \gtrsim f_2$. For a set of points $Y = \{y_k\}$ and an extra element $\tilde{y}$, we denote the collection $\{y_k\} \bigcup \{\tilde{y}\}$ by $\tilde{Y}$, with $||\tilde{Y}||_{l^p} := (||Y||_{l^p} + |\tilde{y}|^p)^{1/p}$ and $||\tilde{Y}||_{l^\infty} := \max(\|Y\|_{l^\infty}, |\tilde{y}|)$. These conventions will be used throughout the paper.

We first review a basic, alternative formulation of $PW_b^p$, $1 \le p \le \infty$. An entire function $f$ is said to be of *exponential type b* if

$$b = \inf\left(\beta : |f(z)| \le e^{\beta |z|}, z \in \mathbb{C}\right).$$

We denote this by writing $\text{type}(f) = b$, and by $\text{type}(f) = \infty$ if $b = \infty$ or $f$ is not entire. By the Paley-Wiener-Schwartz theorem [9], $PW_b^p$ can be equivalently described as the space of all entire functions with $\text{type}(f) \le \pi b$ whose restrictions to $\mathbb{R}$ are in $L^p$. It also follows that $PW_b^p \subset PW_b^q$ for $p < q$. Functions $f \in PW_b^p$ satisfy the classical estimates $\|f'\|_{L^p} \le \pi b \|f\|_{L^p}$ and $\|f(\cdot + ic)\|_{L^p} \le e^{\pi b|c|} \|f\|_{L^p}$, respectively known as the *Bernstein* and *Plancherel-Polya inequalities* [10, 12].

There is a rich and well-developed theory of nonuniform sampling for functions in $PW_b^p$. We only cover a few aspects of it that we will need in this paper, and refer to [12] and [15] for more details. We consider a sequence of points $X = \{x_k\} \subset \mathbb{R}$, indexed so that $x_k < x_{k+1}$. The *separation constant* of $X$ is defined by $\lambda(X) = \inf_k |x_{k+1} - x_k|$, and $X$ is said to be *separated* if $\lambda(X) > 0$. The *generating function* of $X$ is given by the product

$$S(z) = z^{\delta_X} \lim_{r \to \infty} \prod_{0 < |x_k| < r} \left(1 - \frac{z}{x_k}\right), \tag{2}$$

where $\delta_X = 1$ if $0 \in X$ and $\delta_X = 0$ otherwise. For real and separated $X$, such a function $S$ is said to be *sine-type* if the following conditions hold:

(I) The product (2) converges and $\text{type}(S) = \pi b < \infty$.
(II) For any $\varepsilon > 0$, there are positive constants $C_1(\varepsilon)$ and $C_2(\varepsilon)$ such that whenever $\text{dist}(z, X) > \varepsilon$,

$$C_1(\varepsilon) \leq e^{-\pi b|\text{Im}(z)|}|S(z)| \leq C_2(\varepsilon). \tag{3}$$

It can be shown that condition (II) is equivalent to requiring that the bounds (3) only hold in some half plane $\{z : |\text{Im}(z)| \geq c\}$, $c > 0$. Furthermore, a sine-type function $S$ also satisfies the bounds $|S'(x_k)| \approx 1$ and forces $X$ to satisfy $\sup_k |x_{k+1} - x_k| < \infty$ [10].

Now suppose the sequence $X = \{x_k\}$ has a sine-type generating function $S$ with $\text{type}(S) = \pi b$. Let $1 < p < \infty$. Then any $f \in PW_b^p$ can be expressed in terms of its samples $a_k = f(x_k)$,

$$f(z) = \sum_{k=-\infty}^{\infty} a_k \frac{S(z)}{S'(x_k)(z - x_k)}, \tag{4}$$

with uniform convergence on compact subsets of $\mathbb{C}$. Conversely, for any $\{a_k\} \in l^p$, the series (4) converges uniformly on compact subsets of $\mathbb{C}$ and defines a function $f \in PW_b^p$ with $a_k = f(x_k)$ [10].

The simplest example of a sequence $X$ with a sine-type generating function is the uniform sequence $x_k = \frac{k}{b}$, for which $S(z) = \frac{\sin(\pi b z)}{\pi b}$ and the expansion (4) reduces to the WSK sampling theorem. More generally, any finite union of uniform sequences has a sine-type generating function. As a more interesting example, the Bessel function $J_0$ has real, separated zeros, satisfies $J_0(z) = J_0(-z)$, and has the asymptotic formula $J_0(z) = \sqrt{\frac{2}{\pi z}} \cos(z - \frac{\pi}{4})(1 + O(\frac{1}{z}))$ as $|z| \to \infty$ and $|\arg z| < \pi$ (see [14]). This implies that for sufficiently small $\varepsilon > 0$, $S(z) = zJ_0(\frac{\pi z}{2})J_0(\frac{\pi(z+\varepsilon)}{2})$ is a sine-type function with $\text{type}(S) = \pi$. Sequences $X$ with sine-type generating functions are not the most general class for which $f$ has an expansion of the form (4), but they have several convenient properties and cover some important cases encountered in applications, such as that of periodic interpolation points. Such sequences $X$ and various properties of the series (4) have recently been studied in [4] in a computational context.

The above results do not directly carry over to bounded functions $f \in PW_b^\infty$, but in this case we still have the following theorem [2].

**Theorem.** *(Beurling) For a sequence $X = \{x_k\}$, let $N(X, I)$ be the number of $x_k$ in an interval $I$. Then $\|f\|_{L^\infty} \approx \|f(X)\|_{l^\infty}$ for all $f \in PW_b^\infty$ if and only if*

$$D^-(X) := \limsup_{r \to \infty} \inf_a \frac{N(X, [a, a + r))}{r} > b.$$

$D^-(X)$ is called the *lower uniform density* of $X$. For a uniform sequence $x_k = \frac{k}{s}$, $D^-(X) = s$, and Beurling's theorem implies that $f \in PW_b^\infty$ is uniquely determined by its samples if we oversample it beyond its Nyquist rate.

We finally review a few properties of the Banach space $BMO$ of functions with bounded mean oscillation, which has been studied extensively in connection with singular integral operators. It is defined by

$$\left\{f : \|f\|_{BMO} = \sup_I \frac{1}{|I|}\int_I \left| f(t) - \frac{1}{|I|}\int_I f(s)ds \right| dt < \infty \right\},$$

where the supremum runs over all real intervals $I$. The quantity $\|f\|_{BMO}$ is technically a seminorm, since $\|f\|_{BMO}$=$\|f+c\|_{BMO}$ for any constant $c$. Now for any $g \in L^1$, we denote its Hilbert transform by $\mathscr{H}g(z) := \int_{-\infty}^{\infty} \frac{g(t)}{\pi(t-z)}dt$ and its Riesz projections by $\mathscr{P}^{\pm}g := (g \pm i\mathscr{H}g)/2$. We can then consider the "real" Hardy space $H^1(\mathbb{R})$, given by

$$\left\{f : \|f\|_{H^1(\mathbb{R})} = \|f\|_{L^1} + \|\mathscr{H}f\|_{L^1} < \infty \right\}.$$

Finally, it will also be useful to define the subspaces

$$\begin{aligned} \mathbf{U}_1 &= \{f \in C_0^\infty : \int_{-\infty}^{\infty} f(t)dt = 0\} \\ \mathbf{U}_2 &= \{f \in H^1(\mathbb{R}) : (1+t^2)|\mathscr{P}^+ f(t)| \in L^\infty\} \end{aligned}$$

which are both norm dense in $H^1(\mathbb{R})$ [8, 13]. These spaces are all closely related, as the following theorem shows.

**Theorem.** *(Fefferman) $BMO$ is the dual space of $H^1(\mathbb{R})$. More specifically, we have the inequality*

$$\|f\|_{BMO} \approx \sup_{g\in\mathbf{U}} \frac{1}{\|g\|_{H^1(\mathbb{R})}} \left| \int_{-\infty}^{\infty} f(t)\overline{g(t)}dt \right|,$$

*where $\mathbf{U}$ can be taken as $\mathbf{U}_1$ or $\mathbf{U}_2$. Conversely, for any bounded linear functional $L$ on $H^1(\mathbb{R})$, there is an $f \in BMO$ with $\|L\| \approx \|f\|_{BMO}$.*

We write $w = u + iv$ for the complex variable $w$ in what follows. Let $\mathbb{C}^{\pm} = \{w : \pm v > 0\}$ be the upper and lower half planes, and let $P(w,t) = \frac{1}{\pi}\frac{v}{(u-t)^2+v^2}$ be the Poisson kernel on $\mathbb{C}^+$. Now define the square $Q_{a,r} = \{w : a < u < a+r, 0 < v < r\}$. A measure $\mu$ on $\mathbb{C}^+$ is said to be a *Carleson measure* if we have $\mathscr{N}(\mu) := \sup\left(\frac{\mu(Q_{a,r})}{r}, a \in \mathbb{R}, r > 0\right) < \infty$. In other words, the measure $\mu$ of any square protruding from the real axis must be comparable to the length of its edge. The following theorem characterizes $BMO$ in terms of such measures.

**Theorem.** *(Fefferman-Stein) Suppose $\int_{-\infty}^{\infty} \frac{|f(t)|}{t^2+1}dt < \infty$, so that $P(w,\cdot) \star f$ is well-defined. Then*

$$\|f\|_{BMO} \approx \left[\mathscr{N}\left(v\left|\nabla_{u,v}(P(w,\cdot)\star f)\right|^2 dudv\right)\right]^{1/2}. \tag{5}$$

A detailed discussion of $BMO$ and the significance of these theorems can be found in [8] or [13].

# 3 Bandlimited Interpolation of Bounded Data

In this section, we establish a preliminary result showing how adding an extra sample allows us to treat the bandlimited interpolation of bounded data, such as the noise model discussed in Section 1. We define

$$PW_b^+ = \left\{ f \text{ entire} : \limsup_{r\to\infty} \int_{|z|=r} \left| z^{-2} e^{-\pi b|\mathrm{Im}(z)|} f(z) \right| |dz| < \infty \right\}. \tag{6}$$

The Plancherel-Polya inequality shows that $PW_b^\infty \subset PW_b^+$. Functions in $PW_b^+$ can be expanded in the following way.

**Theorem 1.** *Suppose $X = \{x_k\} \subset \mathbb{R}$ is separated and has a sine-type generating function $S$ with* $\mathrm{type}(S) = \pi b$*, and let $\tilde{x} \notin X$. If $f \in PW_b^+$ and $\tilde{A} = f(\tilde{X})$, then*

$$f(z) = \tilde{a}\frac{S(z)}{S(\tilde{x})} + \sum_{k=-\infty}^{\infty} a_k \lim_{z_0\to z} \frac{S(z_0)}{S'(x_k)} \left( \frac{1}{z_0 - x_k} - \frac{1}{\tilde{x} - x_k} \right), \tag{7}$$

*with uniform convergence of compact subsets of $\mathbb{C}$. Conversely, for any $\tilde{A} \in l^\infty$, the series (7) converges uniformly on compact subsets of $\mathbb{C}$ and $f \in PW_b^+$.*

*Proof.* We use a standard complex variable argument. Assume $z$ is in a closed ball $B$ with $z \notin X$, and choose a real sequence $\{r_n\}$ with $r_n \to \infty$ and $\mathrm{dist}(\{r_n\}, X) > 0$. We can then consider the integral

$$J(r_n) := \frac{1}{2\pi i} \int_{|w|=r_n} \frac{f(w)S(z)}{S(w)} \left( \frac{1}{z-w} - \frac{1}{\tilde{x}-w} \right) |dw|.$$

For sufficiently large $n$, it can be seen by calculating residues that

$$J(r_n) = -f(z) + \tilde{a}\frac{S(z)}{S(\tilde{x})} + \sum_{|x_k|<r_n} a_k \frac{S(z)}{S'(x_k)} \left( \frac{1}{z-x_k} - \frac{1}{\tilde{x}-x_k} \right).$$

The inequalities (3) and (6) imply that as $r_n \to \infty$,

$$|J(r_n)| \lesssim \max_{z\in B} |S(z)(z-\tilde{x})| \int_{|w|=r_n} \frac{|f(w)|e^{-\pi b|\mathrm{Im}(w)|}}{|w|^2} |dw| \to 0.$$

By letting $z \to x_k$ for each $x_k \in B$, we obtain the formula (7) for all $z \in B$. For the other direction of Theorem 1, we note that $S$ has simple zeros at exactly $X$, so for $z \in \mathbb{R}$, $|S(z)| \le 2||S'||_{L^\infty} \mathrm{dist}(z, X)$. The Bernstein and Plancherel-Polya inequalities then show that for $z \in \mathbb{C}$ and $d = \sup_k |x_{k+1} - x_k| < \infty$,

$$|S(z)| \lesssim \|S\|_{L^\infty} \min(\mathrm{dist}(z,X), d) e^{\pi b|\mathrm{Im}(z)|}.$$

Now define the sets:

$$\begin{aligned} I_1^w &= (\lfloor \mathrm{Re}(w) \rfloor - \min(1/2, \lambda(X)), \lfloor \mathrm{Re}(w) \rfloor + \min(1/2, \lambda(X))) \\ I_2 &= (-\infty, \lfloor (\mathrm{Re}(z) + \tilde{x})/2 \rfloor) \setminus (I_1^z \bigcup I_1^{\tilde{x}}) \\ I_3 &= (\lfloor (\mathrm{Re}(z) + \tilde{x})/2 \rfloor + 1, \infty) \setminus (I_1^z \bigcup I_1^{\tilde{x}}) \end{aligned}$$

Using the separation of $X$ along with basic properties of lower Riemann sums, we have

$$\begin{aligned}
|f(z)| &\lesssim |\tilde{a}S(z)| + e^{\pi b|\mathrm{Im}(z)|} \|A\|_{l^\infty} \sum_{k=-\infty}^{\infty} \frac{\min(\mathrm{dist}(z,X),d)|z-\tilde{x}|}{|z-x_k||\tilde{x}-x_k|} \\
&\lesssim \|\tilde{A}\|_{l^\infty} e^{\pi b|\mathrm{Im}(z)|} \left(1 + \sum_{k\in\mathbb{Z}\bigcap I_2} \frac{|x_{k+1}-x_k||z-\tilde{x}|}{\lambda(X)|z-x_k||\tilde{x}-x_k|} + \sum_{k\in\mathbb{Z}\bigcap I_3} \frac{|x_k-x_{k-1}||z-\tilde{x}|}{\lambda(X)|z-x_k||\tilde{x}-x_k|}\right) \\
&\lesssim \|\tilde{A}\|_{l^\infty} e^{\pi b|\mathrm{Im}(z)|} \left(1 + \int_{\mathbb{R}\setminus(I_1^z\bigcup I_1^{\tilde{x}})} \frac{|z-\tilde{x}|}{|z-t||\tilde{x}-t|} dt\right) \\
&\lesssim \|\tilde{A}\|_{l^\infty} e^{\pi b|\mathrm{Im}(z)|} (1+\max(\log|z|,0)), \qquad (8)
\end{aligned}$$

which implies that $f \in PW_b^+$. □

This expansion can be compared with the series (4). It is essentially a nonuniform version of the classical Valiron interpolation formula considered in [3], in which the derivative of $f$ at a point is used instead of the extra sample $\tilde{a}$, but the form considered here will be more convenient for our purposes. We also mention that the extra point $\tilde{x}$ plays no special role in the collection $\tilde{X}$, and we isolate it mainly for notational convenience. If we pick any point $x_j \in X$ and let $y_k = x_k$ for $k \neq j$, $y_j = \tilde{x}$ and $\tilde{y} = x_j$, then $\tilde{Y} = \{y_k\} \bigcup \tilde{y}$ satisfies the conditions of Theorem 1 too.

For any $\tilde{A} \in l^\infty$, we call the function $f$ given by (7) the *bandlimited interpolation* of $\tilde{A}$ at $\tilde{X}$. Note that for any given $\tilde{a}_2$ and $\tilde{x}_2 \notin X$, if $g$ is the bandlimited interpolation of $A \bigcup\{\tilde{a}_2\}$ at $X \bigcup \{\tilde{x}_2\}$, then $g(z) = f(z) + cS(z)$ for some constant $c$. Moreover, if $A \in l^2$, then for any given $\tilde{x} \notin X$ we can always choose $\tilde{a}$ so that $f$ coincides with the series (4), or in the special case of uniformly spaced points $X = \{\frac{k}{b}\}$, the usual bandlimited interpolation given by the WSK series (1).

We discuss an example of a $PW_1^+$ function that illustrates many of the typical properties of the series (7). We use the uniform samples $X = \{k\}$ and denote $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$, where $\Gamma$ is the usual gamma function. The properties of $\psi$ are discussed in depth in [14].

**Example.** The function $G_1(z) = \sin(\pi z)\psi(-z)$ is in $PW_1^+ \setminus PW_1^\infty$ and satisfies $a_k = 0$ for $k < 0$ and $a_k = (-1)^k \pi$ for $k \geq 0$.

The function $\psi$ satisfies the estimate

$$\lim_{|z|\to\infty, |\arg z|<\pi} \frac{\psi(z)}{\log z} = 1, \qquad (9)$$

so $G_1$ is not bounded. With $A = \{a_k\}$ given as above, Theorem 1 shows that for any $\tilde{x}$ and $\tilde{a}$, the (unique) bandlimited interpolation of $\tilde{A}$ at $\tilde{X}$ is of the form $G_1(z) + c\sin(\pi z)$. It follows that the samples $A$ have no bandlimited interpolation in $PW_1^\infty$.

It will be instructive to isolate one property of $G_1$ here. A classical formula of Gauss ([14], p. 240) shows that for integer $k > 0$,

$$G_1(k - \frac{1}{2}) = G_1(-k - \frac{1}{2}) = (-1)^k \left(\sum_{m=1}^{k-1} \frac{1}{m} + \sum_{m=k}^{2k-1} \frac{2}{m} + C\right), \qquad (10)$$

so as $z \to \infty$, $|G_1(z)|$ grows logarithmically in between the integer samples. The same applies as $z \to -\infty$, even though the samples at $k < 0$ are all zero. This can be interpreted as a *nonlocal effect*, where the sustained growth of $|G_1|$ on the positive real axis, caused by the "bad behavior" of the samples at $k > 0$, induces growth on the negative real axis too. This property can be seen in the graph of $G_1$ in Figure 1. It is also present in the bandlimited interpolation of Boche and Mönich's example $a_k = 0$, $k < 1$, and $a_k = (-1)^k / \log(k+1)$, $k \geq 1$, where we take $\tilde{x} = \frac{1}{2}$ and $\tilde{a} = 0$.

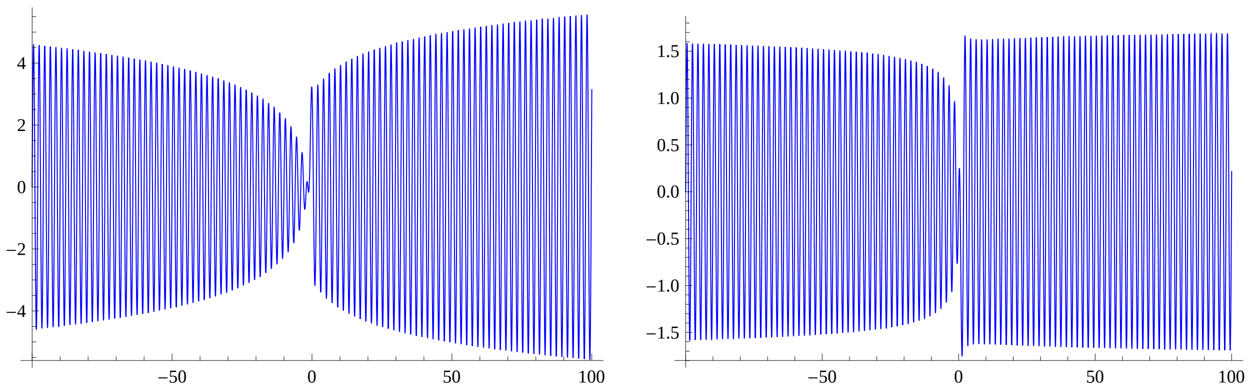



Figure 1: Left: The function $G_1(z)$. Right: The bandlimited interpolation of Boche and Mönich's sequence.

# 4 Bandlimited Interpolation of Random Data

We can now state the main result of this paper.

**Theorem 2.** *Suppose $X \subset \mathbb{R}$ is separated and has a sine-type generating function $S$ with* $\mathrm{type}(S) \leq \pi b$*, and let $\tilde{x} \notin X$. Suppose also that $\tilde{A} = \{a_k\} \bigcup \tilde{a}$ is a collection of i.i.d. random variables uniformly distributed in $[-\alpha, \alpha]$. Let $f$ be the bandlimited interpolation of $\tilde{A}$ at $\tilde{X}$. Then almost surely,*

$$\sup_{r>0} \frac{1}{2r} \int_{-r}^{r} |f(t)| dt < \infty. \tag{11}$$

We make a few comments before proving Theorem 2. This result deals with the same situation discussed in Section 1, even though it has been formulated slightly differently. In the notation of Section 1, we can take $T_k$ to be zero by linearity and only consider the noise $N_k$. As we saw in Section 3, the extra sample $\tilde{a}$ can be taken as deterministic and changed arbitrarily without affecting the result of Theorem 2. The exact probability distribution of $\tilde{A}$ is also of little significance here, and the result holds more generally for any symmetric, finitely supported distribution.

We split the proof of Theorem 2 into three lemmas for clarity. Our approach is to write the function $f$ as the sum of two parts, each with only zero samples in one direction along the real axis, and show that each one is almost surely bounded on that side. This shows directly that the nonlocal effect discussed in Section 3 does not occur. We then move to the deterministic setting and show that this one-sided boundedness forces a certain regularity upon the other side, resulting in the

function having a bounded global average.

For the rest of this section, we assume that $\tilde{X}$ and $S$ are as given in Theorem 2, without repeating the conditions on them every time.

**Lemma 3.** *For $k$ such that $x_k > 0$, let $\{a_k\}$ be a collection of i.i.d. random variables uniformly distributed in $[-\alpha,\alpha]$, let $a_k = 0$ for all other $k$ and let $\tilde{a} = 0$. Suppose $f$ is the bandlimited interpolation of $\tilde{A}$ at $\tilde{X}$. Then $\sup_{t<0}|f(t)| < \infty$ almost surely.*

*Proof.* We can assume that $x_0 = \min(x_k : x_k > 0)$ and $\tilde{x} > 0$, as the general case follows from the remarks after Theorem 1. Let $b_k = \frac{a_k}{S'(x_k)(\tilde{x}-x_k)}$. Then we have

$$\sum_{k=0}^{\infty} E\left(b_k\right) = 0$$

and the separation property shows that for some constant $d$,

$$\begin{aligned} \sum_{k=0}^{\infty} \operatorname{var}\left(b_k\right) &= \frac{\alpha^2}{3}\sum_{k=0}^{\infty}\frac{1}{S'(x_k)^2(\tilde{x}-x_k)^2} \\ &\lesssim \frac{\alpha^2}{3}\sum_{k=0}^{\infty}\frac{1}{(\operatorname{dist}(\tilde{x},X)+\lambda(X)|k-d|)^2} \\ &< \infty. \end{aligned}$$

By Kolmogorov's three-series theorem, $\sum_{k=0}^{\infty} b_k$ converges almost surely. Now let

$$g(t) = \frac{f(t)}{S(t)} = \sum_{k=0}^{\infty}\frac{a_k}{S'(x_k)}\left(\frac{1}{t-x_k} - \frac{1}{\tilde{x}-x_k}\right).$$

It is easy to check that if $\sum_{k=0}^{\infty} b_k$ converges, then $\lim_{t\to-\infty} g(t) = \sum_{k=0}^{\infty} b_k$. Since $|g(0)| < \infty$, it follows by continuity that $\sup_{t<0}|g(t)| < \infty$ almost surely. We also have $\sup_{t<0}|f(t)| \lesssim \sup_{t<0}|g(t)|$, which proves the lemma. $\square$

**Lemma 4.** *For any $\tilde{A} \in l^\infty$, let $f$ be the bandlimited interpolation of $\tilde{A}$ at $\tilde{X}$. Then for each $c > 0$,*

$$\left\|\frac{f(\cdot+ic)}{S(\cdot+ic)}\right\|_{BMO} \lesssim \|A\|_{l^\infty}.$$

*Proof.* Applying Fefferman's duality theorem to the series (7) gives

$$\left\|\frac{f(\cdot+ic)}{S(\cdot+ic)}\right\|_{BMO} \lesssim \sup_{h\in\mathbf{U}_1}\frac{1}{\|h\|_{H^1(\mathbb{R})}}\left|\int_{-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\frac{a_k h(z)}{S'(x_k)}\left(\frac{1}{z+ic-x_k} - \frac{1}{\tilde{x}-x_k}\right)dz\right|.$$

Since $h$ is finitely supported and the series (7) converges uniformly on compact sets, we can interchange the order of summation and integration. $\mathscr{P}^+h$ and $\mathscr{P}^-h$ are in $L^1$, so by analyticity we

have

$$\left\|\frac{f(\cdot+ic)}{S(\cdot+ic)}\right\|_{BMO} \lesssim \sup_{h\in\mathbf{U}_1}\frac{1}{\|h\|_{H^1(\mathbb{R})}}\left|\sum_{k=-\infty}^{\infty}\frac{a_k}{S'(x_k)}\int_{-\infty}^{\infty}\left(\frac{\mathscr{P}^+h(z)+\mathscr{P}^-h(z)}{z+ic-x_k}-\frac{h(z)}{\tilde{x}-x_k}\right)dz\right|$$

$$= \sup_{h\in\mathbf{U}_1}\left|\frac{2\pi i}{\|h\|_{H^1(\mathbb{R})}}\sum_{k=-\infty}^{\infty}\frac{a_k\mathscr{P}^-h(x_k-ic)}{S'(x_k)}\right|$$

$$\lesssim \|A\|_{l^\infty}\sup_{h\in\mathbf{U}_1}\frac{1}{\|h\|_{H^1(\mathbb{R})}}\sum_{k=-\infty}^{\infty}\left|\mathscr{P}^-h(x_k-ic)\right|.$$

Since $X$ is separated, an elementary property of Hardy spaces ([10], p. 138) is that

$$\sum_{k=-\infty}^{\infty}\left|\mathscr{P}^-h(x_k-ic)\right| \lesssim \left\|\mathscr{P}^-h\right\|_{L^1} \le \|h\|_{H^1(\mathbb{R})},$$

which completes the proof. □

**Lemma 5.** *For any $\tilde{A}\in l^\infty$, let $f$ be the bandlimited interpolation of $\tilde{A}$ at $\tilde{X}$. Suppose that $\sup_{t<0}|f(t)|<\infty$ and for some $c>0$, $\frac{f(\cdot+ic)}{S(\cdot+ic)}\in BMO$. Then $\sup_{r>0}\frac{1}{2r}\int_{-r}^{r}|f(t)|dt<\infty$.*

*Proof.* We assume $c=1$ without loss of generality. Let $f^{\pm}(z)=f(z)e^{\pm\pi biz}$, $g(z)=\frac{f(z+i)}{S(z+i)}$, $M_1=\sup_{t<0}|f(t)|$ and $M_2=\sup_{t<0}|g(t)|$. The estimate (8) implies that $\int_{-\infty}^{\infty}\frac{|f(t)|}{t^2+1}<\infty$, so $|f^+|$ has a harmonic majorant on the upper half plane (see [8]) and the reproducing formula $f^+(z)=P(z,\cdot)\star f^+$ holds for $\mathrm{Im}(z)>0$. We can then estimate

$$\begin{aligned}\sup_{t<0}|f^+(t+i)| &\le \sup_{t<0}\left(M_1\int_{-\infty}^{0}P(t+i,s)ds+\int_0^{\infty}|f(s)|P(t+i,s)ds\right)\\ &\le \left(\frac{M_1}{2}+\frac{1}{\pi}\int_0^{\infty}\frac{|f(s)|}{s^2+1}ds\right).\end{aligned}$$

This shows that $M_2<\infty$. Now for any fixed $r>0$,

$$\begin{aligned}\frac{1}{2r}\int_{-r}^{r}|f(t+i)|dt &\lesssim \frac{1}{2r}\int_{-r}^{r}|g(t)|dt\\ &\le \frac{1}{2r}\left(\int_0^r|g(t)|dt-\int_{-r}^{0}|g(t)|dt\right)+M_2\\ &\le \frac{1}{2r}\left(\left|\int_0^r|g(t)|dt-\int_{-r}^{r}|g(s)|ds\right|+\left|\int_{-r}^{0}|g(t)|dt-\int_{-r}^{r}|g(s)|ds\right|\right)+M_2\\ &\le \frac{1}{r}\int_{-r}^{r}\left|g(t)-\frac{1}{2r}\int_{-r}^{r}g(s)ds\right|dt+M_2\\ &\le 2\|g\|_{BMO}+M_2.\end{aligned}$$

We finally use a Poisson integral again to move back to the real line. For $\mathrm{Im}(z)<1$, we have $f^-(z)=P(z-i,\cdot)\star f^-(\cdot+i)$. This gives

$$\frac{1}{2r}\int_{-r}^{r}|f(t)|dt \le e^{\pi b}\frac{1}{2r}\int_{-r}^{r}\int_{-\infty}^{\infty}\frac{|f(s+i)|}{(t-s)^2+1}dsdt$$

$$
\begin{aligned}
&= e^{\pi b}\frac{1}{2\pi r}\int_{-\infty}^{\infty}|f(s+i)|\left(\arctan(r+s)+\arctan(r-s)\right)ds \\
&\leq e^{\pi b}\left(\frac{1}{2r}\int_{-2r}^{2r}|f(s+i)|ds+2\int_{\mathbb{R}\setminus[-2r,2r]}\frac{|f(s+i)|}{s^2+1}ds\right).
\end{aligned}
$$

Taking the estimate (8) into account again, we conclude that

$$
\sup_{r>0}\frac{1}{2r}\int_{-r}^{r}|f(t)|dt<\infty.
$$

□

We can now combine these lemmas to complete the proof.

*Proof of Theorem 2.* For any $\tilde{A}\in l^\infty$, we can write the bandlimited interpolation $f$ of $\tilde{A}$ at $\tilde{X}$ as $f(z)=f_1(z)+f_2(z)+\frac{\tilde{a}S(z)}{S(\tilde{x})}$, where $f_1(x_k)=0$ for $x_k<0$ and $f_2(x_k)=0$ for $x_k\geq 0$. Applying Lemmas 3-5 on $f_1(z)$ and $f_2(-z)$ and noting that $S\in L^\infty$ finishes the proof. □

The statistical incoherence in the samples $\tilde{A}$ in Theorem 2 is the reason we have the bounded average property (11), and it does not generally hold for bounded samples $\tilde{A}$. As an illustration of this, we return to the example function $G_1$ from Section 3 and show that the average of $|G_1(t)|$ is unbounded. It suffices to consider $t<0$. Let $T$ be the tent function

$$
T(t)=\begin{cases} 2t & 0<t\leq\frac{1}{2} \\ 2-2t & \frac{1}{2}<t\leq 1 \\ 0 & \text{otherwise}\end{cases}. \tag{12}
$$

It is clear that $|\sin(\pi t)|\geq\sum_{n=-\infty}^{\infty}T(t+n)$, and the formula (9) implies that $|\psi(t)|\geq\frac{1}{2}\log|t|$ for sufficiently large $t$. This shows that

$$
|G_1(t)|\geq\sum_{n=2}^{\infty}\frac{1}{2}\log(n)T(t+n).
$$

It follows that as $r\to\infty$, $\frac{1}{r}\int_{-r}^{0}|G_1(t)|dt\gtrsim\log r\to\infty$.

Figure 2 below shows an example of the bandlimited interpolation of random data. In the notation of Theorem 2, we use a realization of $\tilde{A}$ with $\alpha=\frac{1}{2}$, and take $x_k=k$ and $\tilde{x}=\frac{1}{2}$. We denote the resulting function by $G_2$. The graphs in Figure 2 can be compared with the functions shown in Figure 1 in Section 3. Unlike those functions, it can be seen that $G_2$ does not steadily grow over long time intervals. Intuitively, this shows how the effect of noisy samples on the bandlimited interpolation is in a sense well-controlled.

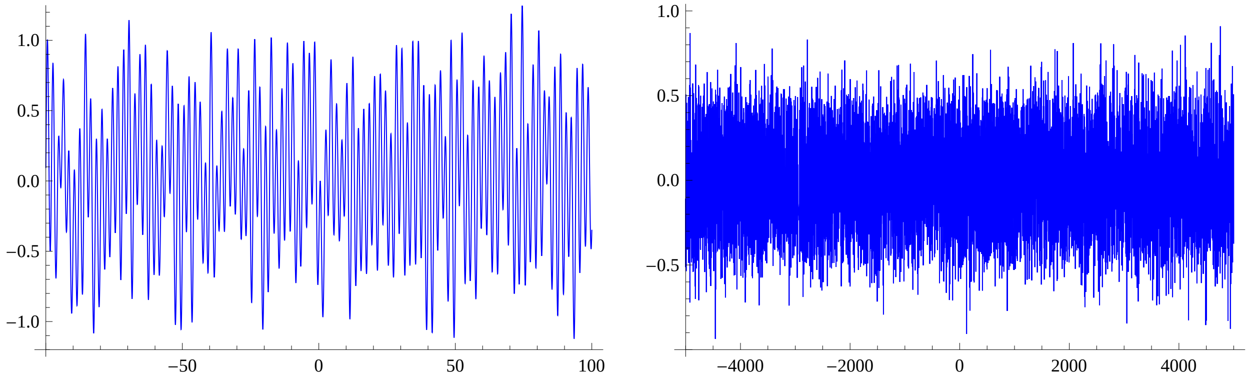



Figure 2: Left: The function $G_2(z)$ on $[-100,100]$. Right: $G_2(z)$ on $[-5000,5000]$.

# 5 Bandlimited BMO Functions

In this section, we study some properties of bandlimited functions in the space $BMO$. Such functions have a somewhat different character than the examples we have seen so far. We fix a point $c$ and define the space $PW_b^\star$ to be the following.

$$PW_b^\star = \{f : \mathrm{type}(f) \leq \pi b, \|f\|_{BMO,c} := |f(c)| + \|f\|_{BMO} < \infty\}$$

The term $|f(c)|$ resolves the ambiguity in the $BMO$ seminorm for constant functions, and $\|f\|_{BMO,c}$ is a (full) norm. It will be shown below that the precise value of $c$ is unimportant and that changing it gives an equivalent norm. Since $f \in BMO$ always satisfies $\int_{-\infty}^{\infty} \frac{|f(t)|}{t^2+1} dt < \infty$ [8], the Paley-Wiener-Schwartz theorem implies that $PW_b^\star \subset PW_b^+$. We first give a version of the Plancherel-Polya inequality for $PW_b^\star$.

**Lemma 6.** *If $f \in PW_b^\star$, then $\|f(\cdot + ic)\|_{BMO} \leq \|f\|_{BMO}\, e^{\pi b|c|}$.*

*Proof.* The proof is similar to the $PW_b^p$ case described in [12]. Define

$$R_\varepsilon^\pm(z) = e^{\mp(\pi b+\varepsilon)\mathrm{Im}(z)} \frac{1}{2r} \int_{-r}^{r} \left| f(z+t) - \frac{1}{2r} \int_{-r}^{r} f(z+s)ds \right| dt,$$

for complex $z$ and real $r$. For each $\varepsilon > 0$, $R_\varepsilon^+$ is a subharmonic function satisfying $|R_\varepsilon^+(z)| \leq \|f\|_{BMO}$ for $z \in \mathbb{R}$ and $\max(\log|R_\varepsilon^+(z)|, 0) \to 0$ as $z \to i\infty$. Applying the Phragmen-Lindelöf principle over $\mathbb{C}^+$gives $|R_\varepsilon^+(z+ic)| \leq \|f\|_{BMO}\, e^{(\pi b+\varepsilon)|c|}$ for $c \geq 0$, and we can repeat the argument with $R_\varepsilon^-$ and $\mathbb{C}^-$ for $c < 0$. Taking the supremum over real $z$ and $r$ and letting $\varepsilon \to 0$ gives the inequality. $\square$

We will now establish several basic properties of $PW_b^\star$.

**Theorem 7.** *Let $f \in PW_b^\star$. Then the following statements hold.*
*I: For each $c \in \mathbb{R}$, $f(\cdot + ic)$ is uniformly Lipschitz continuous on $\mathbb{R}$.*
*II: For any fixed numbers $c$ and $c'$, $\|f\|_{BMO,c} \approx \|f\|_{BMO,c'}$.*
*III: For any given $z \in \mathbb{C}$, the point evaluation functional $z \to f(z)$ is bounded on $PW_b^\star$.*
*IV: $\|f'\|_{L^\infty} \lesssim \|f\|_{BMO}$.*

*Proof.* We set $b = 1$ without loss of generality. We can prove all of the above statements by using the reproducing kernel-like function

$$K(c,t) = \frac{|c|}{\pi t(t-c)} \sin\left(\frac{2\pi N}{c} t\right),$$

where $c \in \mathbb{R}\backslash\{0\}$ and $N$ is any integer greater than $|c|$. As a function of $t$, $K(c,t)$ is entire and satisfies $2\pi \leq \operatorname{type}(K) < \infty$. For any $f \in PW_1^+$,

$$\int_{-\infty}^{\infty} f(t)K(c,t)dt = f(c) - f(0).$$

This can be seen by observing that for $\eta = \pm 1$, the function $\frac{c}{z(z-c)} \exp(2\pi i \eta N z/c) f(z)$ has poles at $c$ and 0 with respective residues $\eta f(c)$ and $-\eta f(0)$. The estimation argument is very similar to the proof of Theorem 1, and we omit the details.

We now suppose that $f \in PW_1^\star$. We want to approximate the $H^1(\mathbb{R})$ norm of $K(c,t) - K(c',t)$, where $c \geq 1$ and $c' \geq 1$. We first integrate the function $\frac{1}{\pi(z-s)} \frac{c}{z(z-c)} \exp(2\pi i \eta N z/c)$, where $s \in \mathbb{R}\backslash\{0, c\}$, and perform the same kind of calculation as before to find that

$$\begin{aligned} \mathscr{H}K(c,s) &= -\frac{1}{\pi s} - \frac{1}{\pi(c-s)} + \frac{c\exp(2\pi i N s/c)}{2\pi s(c-s)} + \frac{c\exp(-2\pi i N s/c)}{2\pi s(c-s)} \\ &= \frac{c\,(\cos(2\pi N s/c) - 1)}{\pi s(c-s)}. \end{aligned}$$

Let $N = \max(\lceil c \rceil, \lceil c' \rceil)$ and define the interval $I^w := [w - \frac{1}{2}, w + \frac{1}{2}]$. We first consider the case where $1 \leq c \leq \frac{3}{2}$ and $|c - c'| > \frac{1}{2}$. Recalling that $T$ is the tent function (12), we have

$$\begin{aligned} &\left\|K(c,\cdot) - K(c',\cdot)\right\|_{L^1} \\ \leq\; & \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \left( \frac{2cT(2Nt/c + n)}{\pi|t(t-c)|} + \frac{2c'T(2Nt/c' + n)}{\pi|t(t-c')|} \right) dt \\ \leq\; & \frac{16N}{\pi} + \int_{\mathbb{R}\backslash(I^0 \bigcup I^c)} \frac{2c}{\pi|t(t-c)|} dt + \int_{\mathbb{R}\backslash(I^0 \bigcup I^{c'})} \frac{2c'}{\pi|t(t-c')|} dt \\ \lesssim\; & |c - c'|. \end{aligned}$$

Now suppose that $1 \leq c \leq \frac{3}{2}$ and $|c - c'| \leq \frac{1}{2}$, so that $N = 2$. Some elementary estimates show that

$$\begin{aligned} &\left\|K(c,\cdot) - K(c',\cdot)\right\|_{L^1} \\ \lesssim\; & \int_{-1/2}^{1/2} \left|\frac{4}{c} - \frac{4}{c'}\right| dt + \int_{1/2}^{5/2} \max\left( \left|\frac{4}{c} - \frac{\sin(4\pi c/c')}{\pi(c-c')}\right|, \left|\frac{4}{c'} - \frac{\sin(4\pi c'/c)}{\pi(c'-c)}\right| \right) dt + \\ & \int_{\mathbb{R}\backslash(-1/2,5/2)} |c - c'||t|^{-3/2} dt \\ \lesssim\; & |c - c'|. \end{aligned}$$

Following the same arguments, we can also obtain the bound $\|\mathscr{H}K(c,\cdot)-\mathscr{H}K(c',\cdot)\|_{L^1} \lesssim |c-c'|$ for the above choices of $c$ and $c'$. By Fefferman's duality theorem and the fact that $K(c,\cdot)-K(c',\cdot) \in \mathbf{U}_2$, we have

$$\|f\|_{BMO} \gtrsim \frac{1}{\|K(c,\cdot)-K(c',\cdot)\|_{H^1(\mathbb{R})}} \left| \int_{-\infty}^{\infty} f(t)(K(c,t)-K(c',t))dt \right| \gtrsim \left| \frac{f(c)-f(c')}{c-c'} \right|, \qquad (13)$$

where the constant in the inequality is independent of $c$ and $c'$. Since the $BMO$ seminorm is translation-invariant, the inequality (13) actually holds for all $c,c' \in \mathbb{R}$. Combining this with Lemma 6 proves (I) and letting $c' \to c$ gives (IV). If we fix $R = |c-c'| > 0$, this also shows that $\|f\|_{BMO} + |f(c)| \gtrsim |f(c')|$, where the implied constant depends on $R$, and we can interchange $c$ and $c'$ to get (II). Finally, the statement (III) is just (II) phrased in a different way. $\square$

*Remark.* The closure of the set of uniformly continuous $BMO$ functions under the $BMO$ seminorm is called $VMO$, for vanishing mean oscillation. Theorem 7 (I) shows that $PW_b^\star \subset VMO$. Note that there are two non-equivalent definitions of $VMO$ in the literature, and we use the one given in [8].

*Remark.* Theorem 7 (IV) is a sharper form of the $p=\infty$ case of Bernstein's inequality. We mention that the opposite inequality does not generally hold (even if $\|f\|_{BMO}$ is replaced by $\|f\|_{BMO,c}$), and there are functions $f$ such that $f' \in PW_b^\infty$ but $f \notin PW_b^\star$.

**Corollary 8.** *Let $f \in PW_b^\star$. Then either $f \in PW_b^\infty$ or there is no separated sequence $X$ with $D^-(X) > 0$ such that $f(X) \in l^\infty$.*

*Proof.* Suppose we have a separated $X = \{x_k\}$ with $D^-(X) > 0$ and $f(X) \in l^\infty$. This means that for some large fixed $r$, every real interval $I$ of length $r$ contains a point $x_n \in X$. Theorem 7 (IV) then shows that for any $t \in I$,

$$|f(t)| = \left| f(x_n) + \int_{x_n}^{t} f'(u)du \right| \lesssim |f(x_n)| + r\|f\|_{BMO}.$$

$\square$

Intuitively, Corollary 8 says that an unbounded $PW_b^\star$ function is large in most places on the real line. It also shows that the bandlimited interpolation of bounded data $\tilde{A} \in l^\infty$ can never be in $PW_b^\star$ unless it is actually in $PW_b^\infty$. This occurs in spite of Lemma 4 and highlights a basic difference between $PW_b^\star$ and $PW_b^p$, $1 < p < \infty$. In Lemma 4, we generally cannot remove the factor $\frac{1}{S(\cdot+ic)}$ from the inequality and conclude that $f \in BMO$. In contrast, for $A \in l^p$, the series (4) can be used to find that $\frac{f(\cdot+ic)}{S(\cdot+ic)} \in L^p$ (see [10]), which clearly implies $f(\cdot+ic) \in L^p$ and thus $f \in L^p$.

We finally study an example of an unbounded $PW_b^\star$ function that illustrates the "largeness" property described above.

**Example.** The function $G_3(z) = \sum_{k=0}^{\infty} (-1)^k \sin\left(\frac{\pi z}{3\cdot 2^k}\right)$ is in $PW_{1/3}^\star \backslash PW_{1/3}^\infty$.

To see this, we use the identity $\sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$ to write $G_3 = G_{3+} + G_{3-}$, where $P(w,\cdot) \star G_{3\pm} = G_{3\pm}(w)$ for $w \in \mathbb{C}^{\pm}$, and then apply the Fefferman-Stein theorem (5) to each part. Let $w = u + iv$. We first note that by analyticity,

$$|\nabla(P(w,\cdot) \star G_{3+})|^2 = |\nabla G_{3+}(u+iv)|^2 = 2|G_{3+}'(w)|^2.$$

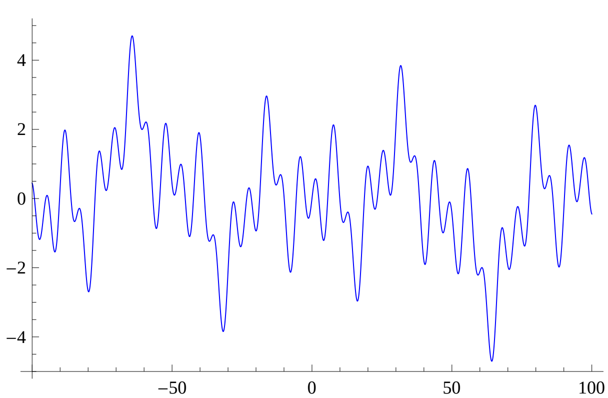
Since $\left|\sin \frac{\pi z}{3\cdot 2^k}\right| \leq \left|\frac{\pi z}{3\cdot 2^k}\right|$ for large $k$, the series defining $G_3$ converges uniformly on compact sets, so we have

$$\begin{aligned}
\mathscr{N}\left(2v\left|G_{3+}'(w)\right|^2 dudv\right) &= \mathscr{N}\left(2v\left|\frac{d}{dw}\frac{1}{2i}\sum_{k=0}^{\infty} e^{\frac{\pi i}{3}2^{-k}w}\right|^2 dudv\right) \\
&\leq \mathscr{N}\left(2ve^{-\frac{2\pi}{3}v}\left(\sum_{k=0}^{\infty}\frac{\pi}{3\cdot 2^k}\right)^2 dudv\right) \\
&\leq 2.
\end{aligned}$$

Doing the same calculation with $G_{3-}$, we find that $G_3 \in PW^{\star}_{1/3}$. On the other hand, $G_3$ satisfies the identity $G_3(2z) = \sin(\frac{2\pi z}{3}) - G_3(z)$. This implies that for integer $n \geq 2$,

$$\begin{aligned}
G_3(2^n) &= (-1)^n g_3(1) + \sum_{k=0}^{n-1}(-1)^{n-k}\sin\left(\frac{\pi 2^k}{3}\right) \\
&= (-1)^n\left(g_3(1) - \frac{\sqrt{3}}{2}(n-2)\right),
\end{aligned}$$

so $G_3 \notin PW^{\infty}_{1/3}$. By Corollary 8, the samples $G_3(X)$ are unbounded for any separated sequence $X$ with $D^-(X) > 0$. It is interesting to note that such a function can still be bounded on a sequence $X$ that is "very sparse" in the sense that $D^-(X) = 0$. It is easy to check that $G_3(3\cdot 2^n) = (-1)^n G_3(3)$ and $G_3(-z) = -G_3(z)$, so $G_3(X) \in l^{\infty}$ for the sequence $x_n = 3\cdot 2^n \mathrm{sign}(n)$. Some graphs of $G_3$ are shown in Figure 3 below.



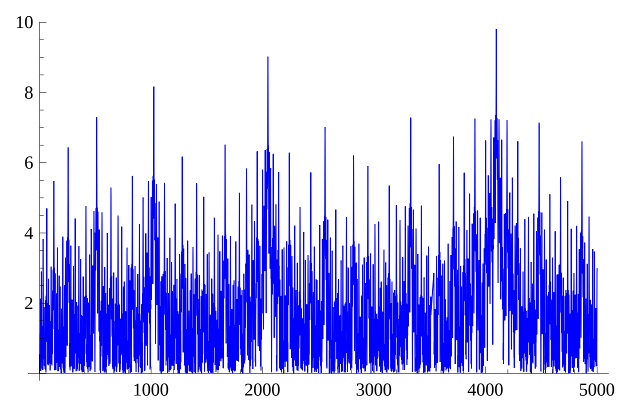




Figure 3: Left: The function $G_3(z)$ on $[-100,100]$. Right: The absolute value of $G_3(z)$ on $[0,5000]$. The peaks at powers of 2 are clearly visible, as well as a self-similarity effect at different scales.

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